Strict inequalities of critical probabilities on Gilbert's continuum percolation graph

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Abstract

Any infinite graph has site and bond percolation critical probabilities satisfying $p_c^{\text{site}} \geq p_c^{\text{bond}}$. The strict version of this inequality holds for many, but not all, infinite graphs.

In this paper, the class of graphs for which the strict inequality holds is extended to a continuum percolation model. In Gilbert's graph with supercritical density on the Euclidian plane, there is almost surely a unique infinite connected component. We show that on this component $p_c^{\text{site}} > p_c^{\text{bond}}$. This also holds in higher dimensions.

1   Introduction

Consider an infinite connected graph G and perform bond percolation by independently marking each edge open with probability p and closed otherwise. The critical probability p_c^{bond} refers to the value of p above which there exists almost surely (a.s.) an infinite connected subgraph of G, of open edges. Similarly, one can perform site percolation by independently marking each vertex of G open with probability p and refer to p_c^{site} as the critical

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probability above which there exists a.s. an infinite connected subgraph of G, of open vertices.

The weak inequality $p_c^{\text{site}} \geq p_c^{\text{bond}}$ can easily be proven by dynamic coupling, see for example Chapter 2 of Franceschetti and Meester (2007). If G is a tree, then it is also easy to see that $p_c^{\text{site}} = p_c^{\text{bond}}$, as each vertex, other than some arbitrarily selected root, can be uniquely identified by an edge and vice versa. By adding finitely many edges to an infinite tree, one can also construct other connected graphs for which the equality holds.

On the other hand, the strict inequality $p_c^{\text{site}} > p_c^{\text{bond}}$ has also been shown to hold in several circumstances. Grimmett and Stacey (1998) proved it for a large class of 'finitely transitive' graphs including the d-dimensional hypercubic lattices.

These graphs, however, do not include the *random* graphs constructed using *continuum* percolation models, because they are not 'finitely transitive': since their average node degree is not bounded, the group action defined by their automorphisms has infinitely many orbits almost surely. These continuum percolation graphs are the focus of this paper. They are of particular interest in the context of communication networks and are treated extensively in the books by Franceschetti and Meester (2007), Meester and Roy (1996), and Penrose (2003).

We consider *Gilbert's graph*, which is defined as follows. Let $\lambda > 0$ and let $\mathscr{P}_\lambda$ be a homogeneous Poisson point process in $\mathbb{R}^2$ of intensity λ. Gilbert's graph, here denoted $G(\mathscr{P}_\lambda, 1)$, has as its vertex set the point set $\mathscr{P}_\lambda$, and the edges are obtained by connecting every pair of points $x, y \in \mathscr{P}_\lambda$ such that $|x - y| \leq 1$, by an undirected edge. It is well known that there exists a critical density value $\lambda_c \in (0, \infty)$, such that if $\lambda > \lambda_c$ then there exists a.s. a unique infinite connected component, while if $\lambda < \lambda_c$ then there is a.s no infinite connected component; see e.g. Meester and Roy (1996). When it exists, we denote this infinite component by $\mathcal{C}$.

In the site percolation model on $\mathcal{C}$, each vertex is independently marked open with probability p, and closed otherwise, and we look for an unbounded connected component in the induced subgraph $\mathcal{C}_v$ of the open vertices. It is easy to see that this is equivalent to rescaling the original Poisson process to one with intensity $p\lambda$ and looking for an unbounded connected component there. It follows that for $\lambda > \lambda_c$ there is a critical value $p_c^{\text{site}} \in (0, 1)$ (namely $p_c^{\text{site}} = \lambda_c/\lambda$) such that if $p > p_c^{\text{site}}$ then there is a.s. an infinite connected component in $\mathcal{C}_v$, and if $p < p_c^{\text{site}}$ then there is a.s. no such infinite component.

In the bond percolation model on $\mathcal{C}$, we independently declare each edge

to be open with probability p, and closed otherwise, and look for an unbounded connected component in the induced subgraph $\mathcal{C}_e$ of the open edges. There is a critical probability p_c^{bond} such that if $p > p_c^{\text{bond}}$ then there is a.s. an infinite connected component in $\mathcal{C}_e$, and if $p < p_c^{\text{bond}}$ then there is a.s. no such infinite component. Observe that $p_c^{\text{bond}} \leq p_c^{\text{site}} < 1$, and it can also be shown by a branching process comparison that $p_c^{\text{bond}} > 0$.

Our main result provides strict inequality between p_c^{site} and p_c^{bond} on Gilbert's graph. Our proof easily extends to 3 or more dimensions.

Theorem 1 *Consider $G(\mathscr{P}_\lambda, 1)$ for $\lambda > \lambda_c$. On $\mathcal{C}$ we have $p_c^{\text{site}} > p_c^{\text{bond}}$.*

The basic strategy is to adapt the enhancement technique developed for percolation on lattices by Menshikov (1987), Aizenman and Grimmett (1991), Grimmett and Stacey (1998). This consists of constructing an 'enhanced' version of the site percolation process for which the critical probability is *strictly less* than that of the original site process. Then one can use dynamic coupling of the enhanced model with bond percolation to complete the proof.

We face two main difficulties when trying to extend the enhancement technique to a continuum random setting. One of these amounts to constructing the desired enhancement on a random graph rather than on a deterministic one. The second one consists in adapting some basic inequalities for the enhanced graph, given in the discrete setting by Aizenman and Grimmett (1991), to the continuum setting. This requires somehow more involved geometric constructions and a careful incremental build-up of the Poisson point process. Once we circumvent these obstacles, it is not too difficult to obtain the final result using a classic dynamic coupling construction.

The enhancement strategy has been proven useful to show strict inequalities in a variety of contexts: Bezuidenhout, Grimmett, and Kesten (1993), and Grimmett (1994), use this technique in the context of Potts and random cluster models; Roy, Sarkar, and White (1998) use it in the context of directed percolation. In the continuum, Sarkar (1997) uses enhancement to demonstrate coexistence of occupied and vacant phases for the three-dimensional Poisson Boolean model. Roy and Tanemura (2002) use it in the context of percolation of different convex shapes.

2 Proof of Theorem 1

We now describe the enhancement needed to prove Theorem 1. Throughout this section we consider Gilbert's graph $G(\mathscr{P}_\lambda, 1)$ with $\lambda > \lambda_c$. The objective is to describe a way to to add open vertices to the site percolation model to make the probability of an infinite cluster bigger, without changing the bond percolation model. To do so, we introduce two kinds of coloured vertices, red vertices (the original open vertices) and green vertices (closed vertices which have been enhanced) and for any two vertices x, y we write that $x \sim y$ if they are joined by an edge. In $G(\mathscr{P}_\lambda, 1)$, if we have vertices x_1, x_2, x_3, x_4, x_5 such that x_1 is closed, has no neighbours other than $x_2, \ldots, x_5$, which are all red, and $x_2 \sim x_3$ and $x_4 \sim x_5$ but there are no other edges amongst x_2, x_3, x_4 and x_5 then we say x_1 is *correctly configured* in $G(\mathscr{P}_\lambda, 1)$, and refer to this as a *bow tie* configuration of edges. If a vertex x is correctly configured we make it green with probability q, independently of everything else; see Figure 1.
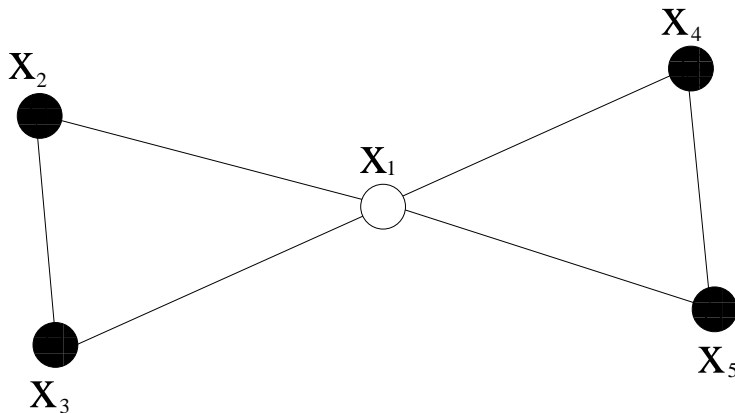

Figure 1: The bow tie enhancement

Let B_n be the open disc of radius n centred at the origin. Let $\underline{Y} = (Y_i, i \geq 0)$ and $\underline{Z} = (Z_i, i \geq 0)$ be sequences of independent uniform $[0, 1]$ random variables. List the vertices of $\mathscr{P}_\lambda$ in order of increasing distance from the origin as $x_1, x_2, x_3, \ldots$. Declare a vertex x_i to be *red* if $Y_i < p$ and *closed* otherwise. Once the sets of red and closed vertices have been decided in this way, apply the enhancement by declaring each closed vertex x_j to be *green* if it is correctly configured and $Z_j < q$. Whenever we insert a vertex of the

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Poisson process at x, it would have values Y_0 and Z_0 associated with it. We shall refer to vertices that are either red or green as being *coloured*.

Let ∂B_n be the annulus $B_n \setminus B_{n-0.5}$ and let A_n be the event that there is a path from a coloured vertex in $B_{0.5}$ to a coloured vertex in ∂B_n in $G(\mathscr{P}_\lambda, 1) \cap B_n$ using only coloured vertices (note that A_n is based on a process completely inside B_n; we do not allow vertices outside of B_n to affect possible enhancements inside B_n).

Let $\theta_n(p, q)$ be the probability that A_n occurs, and define

$$\theta(p, q) \equiv \liminf_{n \to \infty}(\theta_n(p, q)).$$

The following proposition states that $\theta(p, q)$ is indeed the percolation function associated to the enhanced model. From now on we use 'vertex' to refer to a point of the Poisson process and 'point' to refer to an arbitrary location in $\mathbb{R}^2$.

Proposition 1 *There is a.s. an infinite connected component in $G(\mathscr{P}_\lambda, 1)$ using only red and green vertices if and only if $\theta(p, q) > 0$.*

Proof of Proposition 1. For the if part let A'_n be the event that there is a coloured path from $B_{0.5}$ to outside B_{n-2}, so A_n is contained in A'_n. Let $\phi_n(p, q)$ be the probability of A'_n occurring and let $\phi(p, q)$ be the limit as n goes to ∞. Therefore $\phi_n(p, q) \geq \theta_n(p, q)$ for all n so $\phi(p, q) \geq \theta(p, q) > 0$, but $\phi(p, q)$ is just the probability of there being an infinite coloured component intersecting $B_{0.5}$ and it is well known that there is almost surely an infinite coloured component if $\phi(p, q) > 0$.

For the only if part, if there is almost surely an infinite component then $\phi(p, q) > 0$. Given $n \geq 6$, we build up the Poisson process on the whole of B_{n-3}. If there are any closed vertices that are not definitely correctly or incorrectly configured, we build up the process in the rest of their 1-neighbourhood, and this determines whether they are green or uncoloured. If any more closed vertices occur they cannot be correctly configured as they will be joined to a closed vertex. Therefore we have built up the process everywhere in a region R with $B_{n-3} \subset R \subset B_{n-2}$, and all uncoloured vertices at this stage will remain uncoloured. Let V be the set of coloured vertices that are joined by a coloured path to a coloured vertex in $B_{0.5}$ at this stage.

Next, we build out the process radially symmetrically from B_{n-3} (apart from where the process has already been built up) until a vertex v occurs

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that is connected to a vertex in V. Let J be the event that such a vertex v occurs at distance r between $n-3$ and $n-1$ from the origin, so J must occur for A'_n to occur. We can find points $a_1, a_2, \ldots, a_9$ on the line $0v$ extended away from the origin such that a_1 is $r + 0.3$ from the origin, a_2 is $r + 0.6$ from the origin and so on. Surround $a_1, \ldots, a_9$ with circles $D_1, \ldots, D_9$ of radius 0.05 around them. If there is at least one red vertex in each one of these little circles that is contained in B_n when the process continues to the whole of B_n, and v is also red then A_n occurs. Therefore if J occurs then the conditional probability of A_n occuring is at least γ, where

$$\gamma = p(1 - \exp(-0.0025\lambda p\pi))^9,$$

as this is the probability of getting at least one red vertex in each little circle and v being red. Therefore $\theta_n(p, q) \geq \gamma P[J] \geq \gamma\phi(p, q)$ for all $n \geq 6$, so $\theta(p, q) \geq \gamma\phi(p, q) > 0$. $\qquad\square$

Our next lemma provides an analogue of the Margulis-Russo formula for the enhanced continuum model. First, we need to introduce the notion of pivotal vertices.

Given the configuration $(\mathscr{P}_\lambda, \underline{Y}, \underline{Z})$ and inserting a vertex at x we say that x is 1-*pivotal in B_n* if putting $Y_0 = 0$ means that A_n occurs but putting $Y_0 = 1$ means it does not. Notice that x can either complete a path (but it cannot do via being enhanced), or it could make another closed vertex correctly configured which in turn would complete a path. We say that x is 2-*pivotal in B_n* if inserting a vertex at x and putting $Z_0 = 0$ means A_n occurs but putting $Z_0 = 1$ means it does not. That is, $Y_0 > p$ and adding a closed vertex v at x means v is correctly configured and enhancing it to a green vertex means A_n occurs but otherwise it does not.

For $i = 1, 2$ let $E_{n,i}(x)$ be the event that x is i-pivotal in B_n, and set $P_{n,i}(x, p, q) := P[E_{n,i}(x)]$.

Lemma 1 *For all $n > 0.5$ and $p \in (0, 1)$ and $q \in (0, 1)$ it is the case that*

$$\frac{\partial\theta_n(p, q)}{\partial p} = \int_{B_n} \lambda P_{n,1}(x, p, q)\, \mathrm{d}x \tag{1}$$

and

$$\frac{\partial\theta_n(p, q)}{\partial q} = \int_{B_n} \lambda P_{n,2}(x, p, q)\, \mathrm{d}x. \tag{2}$$

Proof. Let $\mathcal{F}$ be the σ-algebra generated by the locations but not the colours of the vertices of $\mathscr{P}_\lambda \cap B_n$. Let N_1 be the number of 1-pivotal vertices. Define $\mathcal{F}$-measurable random variables, $X_{p,q}$ and $Y_{p,q}$ as follows; $X_{p,q}$ is the conditional probability that A_n occurs, and $Y_{p,q}$ is the conditional expectation of N_1, given the configuration of $\mathscr{P}_\lambda$. By the standard version of the Margulis-Russo formula for an increasing event defined on a finite collection of Bernoulli variables (Russo (1981), Lemma 3),

$$\lim_{h \to 0} h^{-1}(X_{p+h,q} - X_{p,q}) = Y_{p,q}, \quad a.s.$$

Let M denote the total number of vertices of $\mathscr{P}_\lambda$ in B_n. By the standard coupling of Bernoulli variables, and Boole's inequality, $|X_{p+h,q} - X_{p,q}| \leq |h|M$ almost surely, and since M is integrable we have by dominated convergence that

$$\frac{\partial \theta_n(p,q)}{\partial p} = \lim_{h \to 0} E[h^{-1}(X_{p+\delta,q} - X_{p,q})] = E[Y_{p,q}] = E[N_1], \tag{3}$$

and by a standard application of the Palm theory of Poisson processes (see e.g. Penrose (2003)), the right hand side of (3) equals the right hand side of (1). The proof of (2) is similar. $\qquad\square$

The key step in proving Theorem 1 is given by the following result.

Lemma 2 *There is a continuous function* $\delta : (0,1)^2 \to (0,\infty)$ *such that for all* $n > 100$, $x \in B_n$ *and* $(p,q) \in (0,1)^2$, *we have*

$$P_{n,2}(x,p,q) \geq \delta(p,q)P_{n,1}(x,p,q). \tag{4}$$

Before proving this, we give a result saying that we can assume there are only red vertices inside an annulus disk of fixed size. For $x \in \mathbb{R}^2$, and $0 \leq \alpha < \beta$, let $C_\alpha(x)$ be the closed circle (i.e., disk) of radius α around x, and let $A_{\alpha,\beta}(x)$ denote the annulus $C_\beta(x) \setminus C_\alpha(x)$. Given n and given $x \in B_n$, let $R_n(x,\alpha,\beta)$ be the event that all vertices in $A_{\alpha,\beta}(x) \cap B_n$ are red.

Lemma 3 *Fix* $\alpha > 3$ *and and* $\beta > \alpha+3$. *There exists a continuous function* $\delta_1 : (0,1)^2 \to (0,\infty)$, *such that for all* $(p,q) \in (0,1)^2$, *all* $n > \beta + 3$ *and all* $x \in B_n$ *with* $|x| < \alpha - 2$ *or* $x > \beta + 2$, *we have*

$$P[E_{n,1}(x) \cap R_n(x,\alpha,\beta)] \geq \delta_1(p,q)P_{n,1}(x).$$

Proof. We shall consider a modified model, which is the same as the enhanced model but with enhancements suppressed for all those vertices lying in $A_{\alpha-1,\beta+1}(x)$. Let $E'_{n,1}(x)$ be the event that x is 1-pivotal in the modified model.

Returning to the original model, we first create the Poisson process of intensity λ in $B_n \setminus A_{\alpha-1,\beta+1}$, and determine which of these vertices are red. Then we build up the Poisson process of intensity λ inside $B_n \cap A_{\alpha-1,\beta+1}$ and for any of these new vertices with more than 4 neighbours, or with at least one closed neighbour outside $A_{\alpha-1,\beta+1}$, we decide whether they are red or closed. This decides whether or not they are coloured as these vertices cannot possibly become green because they are not correctly configured. We now can tell which of the closed vertices outside $A_{\alpha-1,\beta+1}$ are correctly configured, and we determine which of these are green.

This leaves a set W of vertices inside $A_{\alpha-1,\beta+1}$ that have at most four neighbours. If we surround each vertex in W by a circle of radius 0.5 then we cannot have any point covered by more than 5 of these circles as this means that there is a vertex in W with at least 5 neighbours. All of these circles are contained in $C_{\beta+2}$, which has area $\pi(\beta+2)^2$. Therefore

$$|W| \leq \frac{5\pi(\beta+2)^2}{0.5^2\pi} = 20(\beta+2)^2.$$

For x to have any possibility of being 1-pivotal, at this stage there must be a set W' contained in W such that if every vertex in W' is coloured and every vertex in $W \setminus W'$ is uncoloured then x becomes 1-pivotal. In this case, with probability at least $[p(1-p)]^{20(\beta+2)^2}$ we have every vertex in W' red and every vertex in $W \setminus W'$ closed, which would imply event $E'_{n,1}(x)$ occurring. Therefore $P[E'_{n,1}(x)] \geq [p(1-p)]^{20(\beta+2)^2} P[E_{n,1}(x)]$.

Now we note that the occurrence or otherwise of $E'_{n,1}(x)$ is unaffected by the addition or removal of closed vertices in $A_{\alpha,\beta}(x)$. This is because the suppression of enhancements in $A_{\alpha-1,\beta+1}$ means that these added or removed vertices cannot be enhanced themselves, and moreover any vertices they cause to be correctly or incorrectly configured also cannot be enhanced.

Consider creating the marked Poisson process in B_n, with each Poisson point (vertex) x_i marked with the pair (Y_i, Z_i), in two stages. First, add all marked vertices in $B_n \setminus A_{\alpha,\beta}(x)$, and just the red vertices in $B_n \cap A_{\alpha,\beta}(x)$. Secondly, add the closed vertices in $B_n \cap A_{\alpha,\beta}(x)$. The vertices added at the second stage have no bearing on the event $E'_{n,1}(x)$, so $E'_{n,1}(x)$ is independent

of the event that no vertices at all are added in the second stage. Hence,

$$P[E'_{n,1}(x) \cap R_n(x, \alpha, \beta)] \geq \exp(-(1-p)\lambda(\beta^2 - \alpha^2))P[E'_{n,1}(x)],$$

with equality if $|x| \leq n - \beta$.

Finally, we use a similar argument to the initial argument in this proof. Suppose $E'_{n,1}(x) \cap R_n(x, \alpha, \beta)$ occurs. Then there exist at most $20(\beta + 2)^2$ vertices in $A_{\beta,\beta+1}(x) \cup A_{\alpha-1,\alpha}(x)$ which are correctly configured for which the possibility of enhancement has been suppressed. If we now allow these to be possibly enhanced, there is a probability of at least $(1-q)^{20(\beta+2)^2}$ that none of them is enhanced, in which case the set of coloured vertices is the same for the modified model as for the un-modified model and therefore $E_{n,1}(x)$ occurs. Taking

$$\delta_1(p, q) = [p(1-p)(1-q)]^{20(\beta+2)^2} \exp(-(1-p)\lambda(\beta^2 - \alpha^2)),$$

we are done. $\qquad\square$

Proof of Lemma 2. As a start, we fix p and q. We also fix n and $x \in B_n$, and just write $P_{n,i}(x)$ for $P_{n,i}(x, p, q)$. Define event $E_{n,1}(x)$ as before, so that $P_{n,1}(x) = P[E_{n,1}(x)]$. Also, write C_r for the disk $C_r(x)$. For now we assume $30.5 < |x| < n - 30.5$. We create the Poisson process of intensity λ everywhere on B_n except inside C_{30}, and decide which of these vertices are red.

Now we create the process of only the red vertices in $A_{25,30}$ (a Poisson process of intensity $p\lambda$ in this region). Assuming there will be no closed vertices in $A_{25,30}$, we then know which of the closed vertices outside C_{30} are correctly configured, and we determine which of these are green.

Having done all this, let V denote the set of current vertices in $B_n \setminus C_{25}$ that are connected to $B_{0.5}$ at this stage (by connected we mean connected via a coloured path), and let T denote the set of current vertices in $B_n \setminus C_{25}$ that are connected to ∂B_n.

Let $N(V)$ be the 1-neighbourhood of V and let $N(T)$ be the 1-neighbourhood of T. We build up the red process inwards (i.e., towards x from the boundary of C_{25}) on $C_{25} \cap (N(V) \triangle N(T))$ until a red vertex y occurs (if such a vertex occurs). Set $r = |y - x|$. Suppose $y \in N(V)$ (if instead $y \in N(T)$ we would reverse the roles of V and T in the sequel). Then if $T \cap C_{r+0.05} \neq \emptyset$ we say that event F has occurred and we let z denote an arbitrarily chosen

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vertex of $T \cap C_{r+0.05}$. Otherwise, we build up the red process inwards on $C_r \cap N(T) \setminus N(V)$ until a red vertex z occurs (if such a vertex occurs).

Let E_2 be the event that (i) such vertices y and z occur, and (ii) the sets V and T are disjoint, and (iii) $|y - z| > 1$, and (iv) there is no path from y to z through coloured vertices in $B_n \setminus C_{25}$ that are not in $V \cup T$. If $E_{n,1}(x) \cap R_n(x, 20, 30)$ occurs, then E_2 must occur.
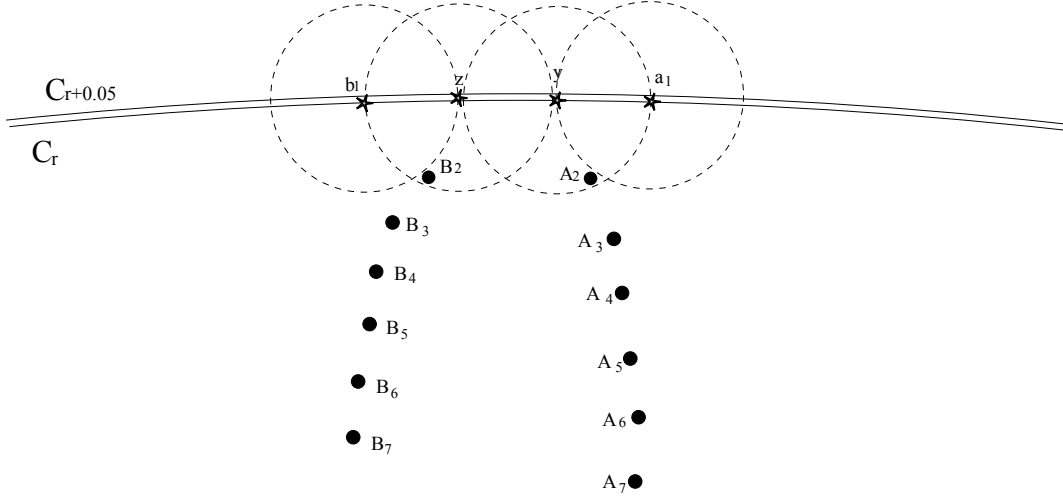

Figure 2: Our convention in the diagrams is to indicate points with lower case letters, and areas with upper case letters. The dashed circles are of radius 1. Here the event F occurs.

Now suppose $E_2 \cap F$ has occurred. Let a_1 be the point (again we use 'point' to refer to a point in $\mathbb{R}^2$) which is at distance r from x and distance 1 from y on the opposite side of the line xy to the side z is on (see Figure 2). Let a_2 be the point lying inside C_r at distance 1.01 from a_1 and 0.99 from y. Let A_2 be the circle of radius 0.005 around a_2. Any red vertex in this circle will be connected to the red vertex y (and therefore to a path to $B_{0.5}$) but cannot be connected to any coloured path to ∂B_n as a_1 is the nearest place for such a vertex to be, given E_2 occurs. Similarly let b_1 be the point lying at distance 1 from z and distance r from x, on the opposite side of xz to y. Then let b_2 be the point at distance 1.01 from b_1 and 0.99 from z, and let B_2 be the circle of radius 0.005 around b_2. Any red vertex in B_2 will be connected to z (and therefore a path to ∂B_n), but not a path to $B_{0.5}$. Also, any vertex in A_2 will be at least 1.1 away from any vertex in B_2.

Now let l be the line through x such that a_2 and b_2 are on different

sides of the line and at equal distance from the line. We can pick points $a_3, a_4, \ldots, a_{30}$ such that a_i is within 0.9 of a_{i+1} for $2 \le i \le 29$, a_{30} and a_{29} are both within 0.9 of x but none of the other a_i's are within 1.1 of x, and none of the $a_i : i \ge 3$ are within 1 of C_r or within 0.5 of l or within 0.01 of another a_j. Do the same on the other side of l with $b_3, b_4, \ldots, b_{30}$. Now consider circles A_i and B_i of radius 0.005 around them. Let I be the event that there is at exactly one red vertex in each of these circles, and also the circles A_2 and B_2, and there are no more new vertices anywhere else in C_{25}, and no closed vertices in $C_{30} \setminus C_{25}$. The probability that I occurs, given $E_2 \cap F$, is at least

$$\delta_2 := (1 - \exp(-0.005^2 \pi \lambda p))^{60} \exp(-900 \pi \lambda).$$

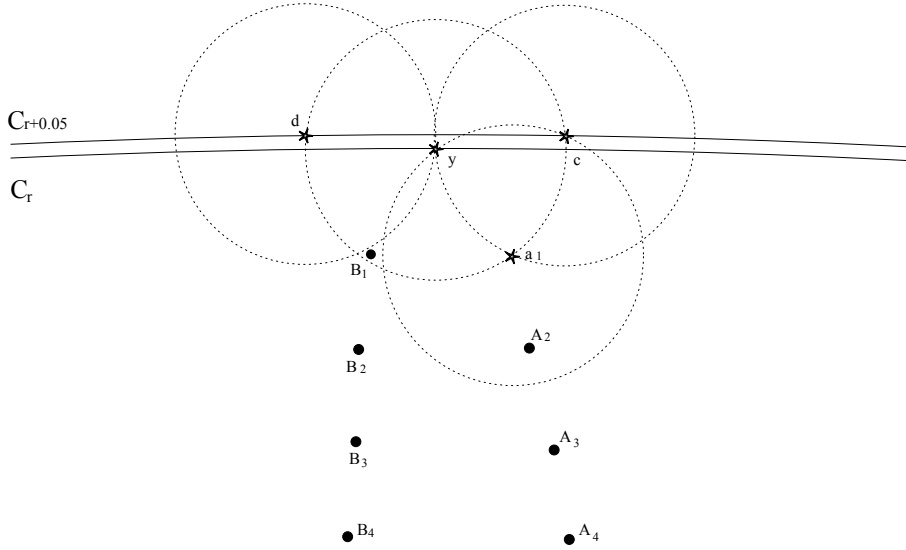If the events E_2, F, I occur and $Y_0 > p$ then x is 2-pivotal.

Figure 3: The case where F does not occur. Here a_1 is the 'worst possible' location for z

Now we consider the case where E_2 occurs but F does not, so z is inside C_r and is connected to a vertex z_1 in T that must be outside $C_{r+0.05}$ as z is the only vertex in T inside $C_{r+0.05}$ (see Figure 3).

Let c be the point at distance 1 from y and $r + 0.05$ from x, on the same side of the line xy as z (assume without loss of generality this is to the right of y). This is the closest z_1 can be. Let a_1 be the point inside C_r at distance 1 from y and 1 from c, so this is the furthest left that z can be. Let d be

the point at distance $r + 0.05$ from x and 1 from y, on the other side of y to c. Then consider the point b_1 inside C_r at distance 1.01 from d and 0.99 from y, and the small circle B_1 of radius 0.005 around b_1. Then any vertex in B_1 is distant at least 1.01 from a_1, and therefore from z, as z cannot be any nearer than a_1. Also any vertex in B_1 will be at least 1.005 from any other vertices in T, as d is the nearest place such a point can be. As before we can then have points $a_2, \ldots, a_{30}$ and $b_2, \ldots, b_{30}$ with small circles around them such that having one red vertex in each of these vertices ensures that x is 2-pivotal. The probability of getting at least 1 red vertex in each of these circles, a red vertex in B_1 and no other new vertices in C_{25}, and no closed vertices in $C_{30} \setminus C_{25}$, is at least δ_2.

So by Lemma 3, the probability that x is 2-pivotal satisfies

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\begin{aligned}
P_{n,2}(x) &\geq \delta_2 P[E_2 \cap F] + \delta_2 P[E_2 \cap F^c] \\
&\geq \delta_2 P[E_{n,1}(x) \cap R_n(x, 20, 30)] \\
&\geq \delta_1 \delta_2 P_{n,1}(x).
\end{aligned}
$$

This proves the claim (4) for the case with $30.5 < |x| < n - 30.5$.

Now suppose $|x| \leq 30.5$. Then we create the Poisson process in $B_n \setminus C_{40}$, and decide which of these vertices are red. Then we create the red process in $A_{39,40}(x)$, and determine which vertices in $B_n \setminus C_{40}$ are green, assuming there are no closed vertices in $A_{39,40}(x)$. We then build up the red process in C_{39} inwards towards x until a vertex y occurs in the process which is connected to ∂B_n. Let H_1 be the event that such a vertex y appears at distance r between 38 and 39 from x, so H_1 must occur for $E_{n,1}(x) \cap R_n(x, 20, 40)$ to occur.

If x is inside $B_{0.5}$ we can choose points a_0 and a_1 such that they are both outside $B_{0.5}$, at distance between 0.8 and 0.9 from x and at distance between 0.1 and 0.2 from each other. We can then choose b_0 and b_1 such that they are both within 0.9 of x, further than 1.5 from a_0 and a_1 and between 0.1 and 0.2 from each other. We can then choose points $a_2, a_3, \ldots, a_{100}$ such that a_i is within 0.9 of a_{i+1} for $1 \leq i \leq 99$, a_{100} is within 0.9 of y, no two a_i are within 0.1 of each other, and no a_i is within 1.1 of x, b_0 or b_1, or inside $B_{0.5}$ for $i \geq 2$. Then consider little circles A_i and B_i of radius 0.05 around these points. If there is at least one red vertex in each of these circles and no vertices anywhere else in C_r then x is 2-pivotal. If x is outside $B_{0.5}$ we choose points in a similar way but make sure b_1 connects with a path to $B_{0.5}$, using little circles $B_2, B_3, \ldots, B_{50}$ which are again of radius 0.05 and are at

least 1.1 from the A_i. Therefore, setting

$$\delta_3 := (1 - \exp(-0.05^2 \pi \lambda p))^{152} \exp(-1600 \pi \lambda)$$

and using Lemma 3, we have for some strictly positive continous $\delta_4(p, q)$ that

$$P_{n,2}(x) \geq \delta_3 P[H_1] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Now suppose $|x| \geq n - 30.5$. In this case the proof is similar. Again, create the Poisson process in $B_n \setminus C_{40}$. Then create the red process in $A_{39,40}(x)$ and determine which vertices in $B_n \setminus C_{40}$ are green, assuming there are no closed vertices in $A_{39,40}(x)$. Then build the red process in $C_{39} \cap B_{n-0.5}$ inwards towards x until a vertex y occurs that is connected to a path of coloured vertices to $B_{0.5}$ but not to ∂B_n. Let H_2 be the event that such a vertex y occurs at distance r between 38 and 39 from x, and that there is no current coloured path from $B_{0.5}$ to ∂B_n, so H_2 has to occur for $E_{n,1}(x) \cap R_n(x, 20, 40)$ to occur. Given this vertex y we can find circles $A_1, A_2, \ldots, A_{100}$ and $B_1, B_2, \ldots, B_{50}$ of radius 0.05 as before such that having a red vertex in each of these little circles but no other vertices in C_r or $\partial B_n \cap C_{40}$ ensures x is 2-pivotal. Therefore in this case

$$P_{n,2}(x) \geq \delta_3 P[H_2] \geq \delta_3 P[E_{n,1}(x) \cap R_n(x, 20, 40)] \geq \delta_3 \delta_4 P_{n,1}(x).$$

Take $\delta(p, q) := \delta_1 \delta_2 \delta_3 \delta_4$. By its construction δ is strictly positive and continuous in p and q, completing the proof of the lemma. $\qquad\square$

The following proposition follows immediately by combining Lemma 1 and Lemma 2.

Proposition 2 *There is a continuous function $\delta : (0,1)^2 \to (0, \infty)$ such that*

$$\frac{\partial \theta_n(p, q)}{\partial q} \geq \delta(p, q) \frac{\partial \theta_n(p, q)}{\partial p}$$

for all $n \geq 100$ and $(p, q) \in (0, 1)^2$.

Proof of Theorem 1. Set $p^* = p_c^{\text{site}}$ and $q^* = (1/8)(p^*)^2$. Then using Proposition 2 and looking at a small box around (p^*, q^*), we can find $\varepsilon \in (0, \min(p^*/2, 1 - p^*))$ and $\kappa \in (0, q^*)$ such that for all $n > 100$ we have

$$\theta_n(p^* + \varepsilon, q^* - \kappa) \leq \theta_n(p^* - \varepsilon, q^* + \kappa).$$

Taking the limit inferior as $n \to \infty$, since θ is monotone in q we get

$$0 < \theta(p^* + \varepsilon, 0) \le \theta(p^* + \varepsilon, q^* - \kappa) \le \theta(p^* - \varepsilon, q^* + \kappa).$$

Now set $p = p^* - \varepsilon$. Then $q^* + \kappa \le p^2$, so that $\theta(p, p^2) > 0$, and by Proposition 1, the enhanced model with parameters (p, p^2) percolates, i.e. has an infinite coloured component, almost surely.

We finish the proof with a coupling argument along the lines of Grimmett and Stacey (1998). Let E be the set of edges and V be the set of vertices of $\mathcal{C}$ (the infinite component). Let $(X_e : e \in E)$ and $(Z_v : v \in V)$ be collections of independent Bernoulli random variables with mean p. From these we construct a new collection $(Y_v : v \in V)$ which constitutes a site percolation process on $\mathcal{C}$. Let $e_0, e_1, \ldots$ be an enumeration of the edges of $\mathcal{C}$ and $v_0, v_1, \ldots$ an enumeration of the vertices. Suppose at some point we have defined $(Y_v : v \in W)$ for some subset W of V. Let $\mathcal{Y}$ be the set of vertices not in W which are adjacent to some currently active vertex (i.e. a vertex $u \in W$ with $Y_u = 1$). If $\mathcal{Y} = \emptyset$ then let y be the first vertex not in W and set $Y_y = Z_y$ and add y to W. If $\mathcal{Y} \ne \emptyset$, we let y be the first vertex in $\mathcal{Y}$ and let y' be the first currently active vertex adjacent to it, then set $Y_y = X_{yy'}$ and add y to W. Repeating this process builds up the entire red site percolation process, if it does not percolate, or a percolating subset of the red site percolation process if it does percolate. In the latter case, the bond process $\{X_e\}$ also percolates.

Now suppose the red site process does not percolate. For any correctly configured vertex x_1 with x_2 up to x_5 as before, x_1 itself is not red. Therefore at most one edge to x_1 has been examined, so we can can find a first unexamined edge (in the enumeration) to x_2 or x_3, and then to x_4 or x_5. We then declare x_1 to be green only if both of these edges are open, which happens with probability p^2. This completes the enhanced site process with $q = p^2$ and every component in this is contained in a component for the bond process $\{X_e\}$.

Therefore, since the enhanced (p, p^2) site process percolates almost surely, so does the bond process, so $p_c^{\text{bond}} \le p < p_c^{\text{site}}$. $\square$

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